Non-Classical 4–Dimensional Minkowski Planes Obtained as Brothers of Semiclassical 4–Dimensional Laguerre Planes

R. LÖWEN

Institut für Analysis
Technische Universität Braunschweig
PockelsstraBe 14
38106 Braunschweig, Germany

G.F. STEINKE

Department of Mathematics and Statistics
University of Canterbury
Private Bag 4800
Christchurch, New Zealand

No. 161 January, 1998

Keywords and phrases. Laguerre plane, Minkowski plane, topological plane, 4–dimensional plane, semiclassical plane, generalized quadrangle, pseudo-oval, Kantor family, 4–gonal family.

This research was supported for the first author by an Erskine Fellowship from the University of Canterbury.

NON-CLASSICAL 4-DIMENSIONAL MINKOWSKI PLANES OBTAINED AS BROTHERS OF SEMICLASSICAL 4-DIMENSIONAL LAGUERRE PLANES

R. Löwen - G.F. Steinke

Institut für Analysis
Technische Universität Braunschweig
Pockelsstraße 14
38106 Braunschweig
Germany

Department of Mathematics
University of Canterbury
Private Bag 4800
Christchurch
New Zealand

Abstract. We describe the first non-classical 4-dimensional Minkowski planes and show that they have 6-dimensional automorphism groups. These planes are obtained by a construction of Schroth [18] from generalized quadrangles associated with the semiclassical 4-dimensional Laguerre planes. All 4-dimensional Minkowski planes that can be obtained in this way from the semiclassical 4-dimensional Laguerre planes are determined.

1. Introduction

In this paper we deal with 4-dimensional Laguerre planes and Minkowski planes and their associated generalized quadrangles. For basic information about 4-dimensional Laguerre planes and Minkowski planes we refer to [27] and for generalized quadrangles to [18]. Whereas there are many examples of non-classical 2-dimensional Minkowski planes, for example see [1] or [14], no non-classical 4-dimensional Minkowski planes were known. The aim of this paper is to construct the first non-classical 4-dimensional Minkowski planes. Our method is based on the unifying

1991 *Mathematics Subject Classification.* 51H15, 51H10.

Key words and phrases. Laguerre plane, Minkowski plane, topological plane, 4–dimensional plane, semiclassical plane, generalized quadrangle, pseudo-oval, Kantor family, 4-gonal family.

This research was supported for the first author by an Erskine Fellowship from the University of Canterbury.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

theory of Schroth [18], who recently discovered a fundamental relationship between 4-dimensional Minkowski planes and Laguerre planes on the one hand and antiregular generalized quadrangles with parameter 2 on the other hand. In sections 2, 3 and 4 we collect general information about 4-dimensional Laguerre planes and Minkowski planes. Moreover, we present some results from Schroth's theory and adapt them to our needs. The key result is that a 4-dimensional Laguerre plane with a certain type of involutory automorphism gives rise to a 4-dimensional Minkowski plane. Section 5 introduces semiclassical Laguerre planes and summarizes the information needed on their automorphism groups. We further determine all involutory automorphisms of the semiclassical Laguerre planes that lead to 4-dimensional Minkowski planes and show that the non-classical Minkowski planes obtained have 6-dimensional automorphism groups that fix a distinguished point. In section 6, the construction of Minkowski planes is made explicit. We also show that the derived affine plane at the distinguished point of such a non-classical Minkowski plane is a 'semiclassical' translation plane admitting an irreducible action of $\mathrm{SL}_2\mathbb{R}$; in particular, this affine plane has an 8-dimensional automorphism group. A somewhat different approach to our construction of Minkowski planes is presented in section 7. It uses the notion of pseudo-oval developed in [11]; a pseudo-oval encodes the information needed for the construction of an elation Laguerre plane or an elation generalized quadrangle in a particularly neat fashion.

2. Minkowski planes

A *Minkowski plane* $\mathcal{M} = (P, \mathcal{K}, \|_+, \|_-)$ consists of a set of points P, a set of at least two circles $\mathcal{K}$ and two equivalence relations $\|_+$ and $\|_-$ on P (parallelisms). We always consider circles as subsets of P. Furthermore it satisfies the following axioms.

(i) three mutually non-parallel points (that is, neither (+)-parallel nor (−)-parallel) can be joined by a unique circle,

(ii) the circles which touch a fixed circle K at $p \in K$ partition $P \setminus |p|$ (here $|p| = |p|_+ \cup |p|_-$ denotes the union of the two parallel classes of p, and two circles are said to *touch* if they are identical or their intersection is a single point),

(iii) each parallel class meets each circle in a unique point (parallel projection),

(iv) each (+)-parallel class and each (−)-parallel class intersect in a unique point, and

(v) there is a circle that contains at least three points.

Sometimes we shall specify the two parallelisms of a Minkowski plane by writing down the total set of all equivalence classes; this is clearly sufficient. When we consider automorphisms, we require that parallelism is invariant, but the two parallelisms may be interchanged.

Following Schenkel [14] we define a *topological Minkowski plane* as a Minkowski plane in which the point set P, the circle set $\mathcal{K}$ and the two sets of parallel classes carry topologies such that the operations of joining, touching, intersecting circles, intersecting parallel classes and circles and intersecting parallel classes of different types are continuous on their domains of definition. A topological Minkowski plane is called (locally) compact, connected, or finite-dimensional if the point space has the respective topological property. For brevity, a locally compact connected finite-dimensional topological Minkowski plane will be called a *finite-dimensional Minkowski plane.* According to [10, 2.3] a finite-dimensional Minkowski plane can only be of dimension 2 or 4.

The classical model of a 2- or 4-dimensional Minkowski plane is obtained as the geometry of non-trivial plane sections of a ruled quadric in the real or complex projective 3-dimensional space respectively. In these cases the topologies on the point set and on the set of circles are induced from the surrounding projective 3-space (the set of planes in the projective 3-space carries a natural topology which can be obtained by duality from the topology on the point set in the 3-space).

Every parallel class and every circle of a 4-dimensional Minkowski plane is homeomorphic to the 2-sphere $\mathbb{S}_2$, and the point space is homeomorphic to $\mathbb{S}_2 \times \mathbb{S}_2$. The homeomorphism can be chosen such that the two parallelisms are the canonical ones on $\mathbb{S}_2 \times \mathbb{S}_2$ given by equality in the first and second coordinates respectively.

Associated with every point p of $\mathcal{M}$ there is a derived incidence structure, called the *derived affine plane* $\mathcal{A}_p = (A_p, \mathcal{G}_p)$ at p, whose point set A_p consists of all points of $\mathcal{M}$ that are not parallel to p and whose set of lines $\mathcal{G}_p$ consists of all restrictions to A_p of circles of $\mathcal{M}$ passing through p and of parallel classes not passing through p. Indeed, $\mathcal{M}$ is a Minkowski plane if and only if all incidence structures $\mathcal{A}_p$ are affine planes. It can easily be seen that derived affine planes of the classical Minkowski plane are even topological locally compact connected affine planes. More generally, it was shown in [22] that this is true for each finite-dimensional Minkowski plane. Furthermore, the projective extension $\mathcal{P}_p$ of $\mathcal{A}_p$ becomes a topological locally compact connected projective plane.

According to [10, 2.5] the classical complex Minkowski plane can be characterized in terms of a single derivation. A 4-dimensional Minkowski plane is isomorphic to the classical complex Minkowski plane if and only if at least one derived affine plane is Desarguesian. This is due to a remarkable result of T. Buchanan [4] which says that the only topological ovals in the (Desarguesian) complex projective plane are the non-degenerate conics.

The automorphism group of a 4-dimensional Minkowski plane $\mathcal{M}$ is a Lie group of dimension at most 12 with respect to the compact-open topology, see [20]. It was shown in [25] that a 4-dimensional Minkowski plane that admits an at least 8-dimensional automorphism group must be classical. In [26] it was proved that a non-classical 4-dimensional Minkowski plane admitting a 7-dimensional automorphism group has a unique point that is fixed by the given group and that the derivation $\mathcal{A}_p$ at this point is a 4-dimensional translation plane with an exactly 7-dimensional (full) collineation group. Using the classification of 4-dimensional translation planes admitting 7-dimensional collineation groups in [3], it is, in principle, possible to classify all 4-dimensional Minkowski planes admitting a 7-dimensional automorphism group. In practice however, the incidence axioms of a Minkowski plane are hard to verify. In this note we therefore follow a different path in order to obtain 4-dimensional Minkowski planes. Our examples will have 6-dimensional automorphism groups, and the collineation group of $\mathcal{A}_p$ will be 8-dimensional. We arrive at these examples by exploiting the following result of Schroth (for generalized quadrangles and the notion of parameter see the next section).

Theorem 1 ([18, Theorem 5.37]). *Let $\mathcal{Q} = (P, \mathcal{L})$ be a compact antiregular generalized quadrangle with parameter 2 and let τ be an involutory collineation such that the space X of fixed points and the space $\mathcal{X}$ of fixed lines form a compact subquadrangle with parameters $(2, 0)$. Then*

$$\mathcal{M}(\mathcal{Q}, \tau) = (X, \{X \cap q^\perp \mid q \in P \setminus X\}, \mathcal{X}, \in)$$

is a 4-dimensional Minkowski plane. In fact, every 4-dimensional Minkowski plane arises in this way.

3. Laguerre planes and generalized quadrangles

A *Laguerre plane* $\mathcal{L} = (P, \mathcal{C}, ||)$ is an incidence structure consisting of a point set P, a circle set $\mathcal{C}$ and an equivalence relation $||$ (parallelism) defined on the point set such that three mutually non-parallel points can be joined by a unique circle, such that the circles which touch a fixed circle K at $p \in K$ partition $P \setminus |p|$, such that each parallel class meets each circle in a unique point (parallel projection), and such that there is a circle that contains at least three points (compare [7] and [8]).

We are interested in Laguerre planes that are *topological*; this means that the sets of points, circles and parallel classes carry topologies and that the operations of joining, touching, intersecting circles and intersecting parallel classes and circles are continuous on their domains of definition. We assume throughout that the topologies are locally compact, connected and of finite dimension. Then the dimension is necessarily 2 or 4, according to [5].

The classical 2- or 4-dimensional Laguerre plane is obtained as the geometry of non-trivial plane sections of an elliptic cone in real or complex projective 3-dimensional space, respectively, with its vertex removed.

There are many models of 2-dimensional Laguerre planes known; for example, [12] presents a large class of planes. At present, only one family of 4-dimensional Laguerre planes and two single planes are known explicitly. However, their sisterhoods in the sense of Schroth (see [18, Theorem 6.1] or [19]) provide many further examples. Circles and parallel classes of 4-dimensional Laguerre planes are homeomorphic to the 2-sphere $\mathbb{S}_2$ and to $\mathbb{R}^2$ respectively. The point sets of all 4-dimensional Laguerre planes are 4-dimensional manifolds homeomorphic to the tangent bundle of the 2-sphere.

The *Lie geometry* associated with a Laguerre plane $\mathcal{L}$ has point set consisting of the points of $\mathcal{L}$ plus the circles of $\mathcal{L}$ plus one additional point at infinity, denoted by ∞. Lines of the Lie geometry are the parallel classes of $\mathcal{L}$ extended by ∞ and the extended tangent pencils of $\mathcal{L}$, that is, collections of the form

$$\{D \mid D \text{ touches } C \text{ at } x\} \cup \{x\}$$

for a circle C and a point $x \in C$. Incidence is the natural one. It is proved in the work of Schroth [18] that the Lie geometry of a 4-dimensional Laguerre plane is an antiregular topological generalized quadrangle in the sense to be defined next.

First, a *generalized quadrangle* is an incidence geometry $\mathcal{Q} = (Q, \mathcal{G})$ such that every point $p \in Q$ is on at least two lines and dually, every line $L \in \mathcal{G}$ carries at least two points, and such that the following condition is satisfied: for each anti-flag (p, L), $p \notin L$, there is a unique flag (q, M), $q \in M$, such that $q \in L$ and $p \in M$. The classical generalized quadrangles are associated with classical groups. Quadrangles in general form a particular class of buildings.

For a point p, we denote by $p^{\perp}$ the set of all points that can be joined to p by a line, including p itself. An *antiregular* generalized quadrangle is a generalized quadrangle such that for any set $\{p, q, r\}$ of three mutually non-collinear points, the intersection $p^{\perp} \cap q^{\perp} \cap r^{\perp}$ is either empty or contains precisely two points. For each point p of an antiregular generalized quadrangle the *derivation at* p is the incidence structure $(p^{\perp} \setminus \{p\}, \{p^{\perp} \cap q^{\perp} \mid q \in P \setminus p^{\perp}\})$ with parallelity of points being the collinearity with p in the generalized quadrangle.

A *topological generalized quadrangle* is a generalized quadrangle where the point set and the set of lines carry Hausdorff topologies such that the mapping that takes an anti-flag (p, L) to the unique flag (q, M) where $p \in M$ and $q \in L$ is continuous. A topological generalized quadrangle is called *compact* if the point set and the set of lines carry compact topologies.

The dual of a topological generalized quadrangle, i.e., the roles of points and lines are exchanged, is a topological generalized quadrangle too. Hence every general topological result about the point set or about lines (considered as point rows) is equally valid for the set of lines and for line pencils respectively. Forst [6] first investigated thick topological generalized quadrangles; thickness means that pencils and lines have at least three elements. In particular, he showed that any two lines are homeomorphic, that the point set is locally homeomorphic to the product of two lines and a line pencil, and dually. Later Grundhöfer-Knarr [9] studied non-discrete, locally compact generalized quadrangles and showed with methods first developed for stable planes that if lines are connected and at most 2-dimensional, then in fact lines are homeomorphic to spheres.

A compact generalized quadrangle with *(topological) parameters* (s, t) is a generalized quadrangle where all lines are homeomorphic to the s-dimensional sphere $\mathbb{S}_s$ and all line pencils are homeomorphic to the t-dimensional sphere $\mathbb{S}_t$. Note that the 0-sphere consists of 2 points. If $s = t$ we say that the generalized quadrangle has parameter s rather than parameters (s, s).

The point set $\{\infty\} \cup P \cup \mathcal{K}$ of the Lie geometry obtained from a 4-dimensional Laguerre plane $\mathcal{L} = (P, \mathcal{K}, \|)$ carries a compact topology such that the topologies induced on P and on $\mathcal{K}$ are the given ones and such that $\mathcal{K}$ is open. Indeed Schroth proved the following important result ([16] and [17], see also [18, Theorem 3.13]).

Theorem 2. *The associated Lie geometry of every* 4*-dimensional Laguerre plane with respect to certain natural topologies is a compact antiregular generalized quadrangle with parameter* 2*.*

Conversely, each derivation of a compact antiregular generalized quadrangle with *parameter* 2 *is a* 4*-dimensional Laguerre plane.*

Another remarkable result of Schroth [15] even says that every compact generalized quadrangle with parameter $k \in \{1, 2\}$ can be constructed from a $2k$-dimensional Laguerre plane: either the derivation at every point of the quadrangle, or the derivation at every point of the dual quadrangle, yields a $2k$-dimensional Laguerre plane. Equivalently, every generalized quadrangle with parameter $k \in \{1, 2\}$ is antiregular up to duality.

4. The view from within a Laguerre plane

How can we use the results of Schroth to construct 4-dimensional Minkowski planes? By Theorem 1, every such plane $\mathcal{M}$ can be obtained from an involutory automorphism τ of some antiregular compact quadrangle $\mathcal{Q}$ with parameter 2; in fact, $\mathcal{Q}$ can be taken to be the lifted Lie geometry of $\mathcal{M}$ (which we have not defined). Now the derivation of $\mathcal{Q}$ at an arbitrary fixed point p of τ is a 4-dimensional Laguerre plane $\mathcal{L}$, and τ induces an automorphism of $\mathcal{L}$ which determines τ completely and will again be denoted by τ. We could have started this construction from the pair $(\mathcal{L}, \tau)$, defining $\mathcal{Q}$ as the Lie geometry of $\mathcal{L}$. We call $\mathcal{M}$ the *brother of* $\mathcal{L}$ *with respect to* τ in this situation, and we write $\mathcal{M} = \mathcal{M}(\mathcal{L}, \tau)$. Our first aim is to characterize the pairs $(\mathcal{L}, \tau)$ for which this construction works, i.e., those pairs where the extension of τ to the Lie geometry $\mathcal{Q}$ satisfies the condition of Theorem 1. Our result sharpens [18, Theorem 6.4].

Theorem 3. *Let $(\mathcal{L}, \tau)$ be a pair consisting of a* 4*-dimensional Laguerre plane $\mathcal{L}$ and an involutory automorphism τ of $\mathcal{L}$. Such a pair defines a brother of $\mathcal{L}$ which is a* 4*-dimensional Minkowski plane if and only if τ fixes precisely the points of two parallel classes of $\mathcal{L}$.*

Proof. Suppose that the construction of a brother works for the pair $(\mathcal{L}, \tau)$. Then, with the notation introduced above, $\mathcal{L}$ is the derivation of $\mathcal{Q}$ at some point p; hence, the point set of $\mathcal{L}$ is $p^\perp \setminus \{p\}$, and the parallel classes are the lines of $\mathcal{Q}$ passing through p. The condition of Theorem 1 requires that the fixed point set of τ on $\mathcal{Q}$ is a subquadrangle with parameters $(2, 0)$, that is, it should contain exactly two lines of $\mathcal{Q}$ through each fixed point. The two such lines through p are the required parallel classes whose union is the fixed point set of τ on $\mathcal{L}$.

Conversely, assume that τ fixes precisely the points of two parallel classes π_1, π_2 of $\mathcal{L}$. In order to apply Theorem 1 to the extension of τ on $\mathcal{Q}$, we have to show that its fixed elements form a subquadrangle with parameters (2,0).

Let $p_1 \in \pi_1$ and $p_2 \in \pi_2$ be two points. In the derived affine plane of $\mathcal{L}$ at p_1 the automorphism τ induces a central collineation with axis π_2 and centre on the infinite line. Suppose that the centre is the infinite point of π_2. Then τ fixes every parallel class (as a set) and acts on each as a homeomorphism of order at most 2. Since each parallel class is homeomorphic to $\mathbb{R}^2$, this implies that τ fixes at least one point on each parallel class. Hence, in this case, $\tau = \mathrm{id}$ — a contradiction. This shows that the centre cannot be the infinite point of π_2. Hence the central line through p_2 comes from a circle in $\mathcal{L}$. Therefore there is precisely one circle fixed by τ through any pair of points one each on π_1 and π_2. Moreover, two fixed circles that have a fixed point (say p_1) in common are touching circles (i.e., parallel lines in $\mathcal{A}_{p_1}$).

It follows that the fixed elements form a subquadrangle, i.e., none of the lines needed to satisfy the definition of a quadrangle is lost in the substructure. This subquadrangle has exactly two lines through each of its points; indeed, p belongs to π_1 and π_2, a fixed point belongs to one parallel class and one touching pencil, and a fixed circle belongs to two touching pencils. Moreover, every point row is a 2-sphere, hence the subquadrangle has parameters $(2,0)$. Now Theorem 1 shows that $\mathcal{M}(\mathcal{L},\tau)$ is a 4-dimensional Minkowski plane; compare also [18, Theorem 6.4]. $\square$

We now use Theorem 1 together with the facts obtained in the last proof to give an explicit description of the brother $\mathcal{M} = \mathcal{M}(\mathcal{L},\tau)$ in terms of $\mathcal{L}$. Let π_1 and π_2 be the two parallel classes that are fixed pointwise and let $\overline{\pi}_1 = \pi_1 \cup \{\infty_1\}$ and $\overline{\pi}_2 = \pi_2 \cup \{\infty_2\}$ be their respective one-point-compactifications. Let $\mathcal{F}$ be the collection of all circles of $\mathcal{L}$ fixed by τ.

The point set of $\mathcal{M}$ is the fixed point set X of τ in $\mathcal{Q}$. From the point of view of $\mathcal{L}$, the elements of X are the point p, the points of π_1 and π_2, and the fixed circles $K \in \mathcal{F}$. Since a circle in $\mathcal{F}$ is uniqely determined by its intersection with π_1 and π_2, we can use $\overline{\pi}_1 \times \overline{\pi}_2$ to coordinatise X. We use (∞_1, ∞_2) to represent p and identify $w \in \pi_2$ with (∞_1, w) and, likewise, $z \in \pi_1$ with (z, ∞_2). The topology of X is the product topology; this is clear for the subset $\pi_1 \times \pi_2$, and we shall not use more than that.

The parallel classes of $\mathcal{M}$ are the lines fixed by τ in $\mathcal{Q}$. Since lines in $\mathcal{Q}$ are either parallel classes or touching pencils of $\mathcal{L}$, we obtain two classes through p given by $\{(\infty_1, w) \mid w \in \overline{\pi}_2\}$ and $\{(z, \infty_2) \mid z \in \overline{\pi}_1\}$. Now we consider a fixed touching pencil L. It contains a unique point, which must be a fixed point $z \in \pi_1$, say. The pencil L contains a unique circle K_w through each point $w \in \pi_2$, but $\tau(K_w)$ passes through the same point, hence $K_w \in X$. This is the point of $\mathcal{M}$ represented by (z, w), and we see that $L = \{z\} \times \bar{\pi}_2$. Thus the parallel classes in $\bar{\pi}_1 \times \bar{\pi}_2$ are precisely the fibers of the cartesian product. We say that the two parallelisms on $\bar{\pi}_1 \times \bar{\pi}_2$ are the canonical ones. Incidentally, this shows again that the fixed elements of τ in $\mathcal{Q}$ form a quadrangle (of rather trivial type).

The circles of $\mathcal{M}$ are given by Theorem 1 as the sets $q^\perp \cap X$, where q is a point of $\mathcal{Q}$ not fixed by τ. If q comes from a point of the Laguerre plane, then $q^\perp \cap X$ consists of (∞_1, ∞_2) and (the coordinates of) all circles in $\mathcal{F}$ through q. If, however, q comes from a circle C of the Laguerre plane, then $q^\perp \cap X$ consists of all the circles in $\mathcal{F}$ that touch C at a point not on $\pi_1 \cup \pi_2$ and the two points $C \cap \pi_1$ and $C \cap \pi_2$. Note that there is no circle in $\mathcal{F}$ that touches C at $C \cap \pi_1$ or $C \cap \pi_2$. We shall often simply write $q^\perp$ instead of $q^\perp \cap X$.

In summary we obtain the following.

Theorem 4. *A brother Minkowski plane $\mathcal{M}(\mathcal{L},\tau)$ as obtained in Theorem 3 may be described as follows. Let $\overline{\pi}_1 = \pi_1 \cup \{\infty_1\}$ and $\overline{\pi}_2 = \pi_2 \cup \{\infty_2\}$ be the one-point-compactifications of the parallel classes π_1 and π_2, respectively. The point set of*

$\mathcal{M}(\mathcal{L},\tau)$ is $\overline{\pi}_1 \times \overline{\pi}_2$. The parallelisms are given by $(z,w) \parallel_+ (z',w')$ if and only if $z = z'$ and $(z,w) \parallel_- (z',w')$ if and only if $w = w'$. Let $\mathcal{F}$ be the collection of all circles fixed by τ, let $\mathcal{F}_p$ be the collection of all circles fixed by τ that pass through the point p and let $\mathcal{F}_C$ be the collection of all circles fixed by τ that touch the circle C. Then circles of $\mathcal{M}(\mathcal{L},\tau)$ are of the form

$$p^\perp = \{(\infty_1,\infty_2)\} \cup \{(C\cap\pi_1, C\cap\pi_2) \mid C \in \mathcal{F}_p\}$$

for $p \in P \setminus (\pi_1 \cup \pi_2)$ and

$$C^\perp = \{(C'\cap\pi_1, C'\cap\pi_2) \mid C' \in \mathcal{F}_C\} \cup \{(\infty_1, C\cap\pi_2), (C\cap\pi_1, \infty_2)\}$$

for $C \in \mathcal{C} \setminus \mathcal{F}$. $\square$

5. Semiclassical Laguerre planes and their involutory automorphisms

5.1 The planes. We describe the *semiclassical Laguerre planes* as constructed in [23]. For $q \in \mathbb{R}$, $0 \le q < 1$ let $f_q: \mathbb{C} \to \mathbb{C}$ be the mapping defined by $f_q(z) = z + q\bar{z}$ where $\bar{z}$ denotes the complex conjugate of z. (More generally, for $a, b \in \mathbb{C}$ the $\mathbb{R}$-linear mapping defined by $f_{a,b}(z) = az + b\bar{z}$ is bijective if and only if $\|a\| \ne \|b\|$; in this case the inverse is given by $f_{a,b}^{-1}(z) = \frac{1}{\|a\|^2 - \|b\|^2}(\bar{a}z - b\bar{z})$.) The point set of the semiclassical Laguerre plane $\mathcal{L}_q$ is

$$P = (\mathbb{C} \cup \{\infty\}) \times \mathbb{C}.$$

The subset $\mathbb{C} \times \mathbb{C}$ carries the product topology; this is not true for P itself. Two points (z,w) and (z',w') are parallel if and only if $z = z'$. Circles of $\mathcal{L}_q$ are of the form

$$\begin{aligned} K_{a_2,a_1,a_0} = {} & \{(z,w) \in \mathbb{C}\times\mathbb{C} \mid w = a_2z^2 + a_1z + a_0, \operatorname{Im}(z) \ge 0\} \\ & \cup \{(z,w) \in \mathbb{C}\times\mathbb{C} \mid w = f_q^{-1}\big(f_q(a_2)z^2 + f_q(a_1)z + f_q(a_0)\big), \operatorname{Im}(z) \le 0\} \\ & \cup \{(\infty, a_2)\} \end{aligned}$$

for $a_0, a_1, a_2 \in \mathbb{C}$. Clearly, we obtain the classical complex Laguerre plane for $q = 0$.

We call these planes semiclassical since on each halfplane $P_+ = \{(z,w) \in P \mid \operatorname{Im}(z) \ge 0 \text{ or } z = \infty\}$ and $P_- = \{(z,w) \in P \mid \operatorname{Im}(z) \le 0 \text{ or } z = \infty\}$ the geometry and topology of $\mathcal{L}_0$ is induced. The two halves are glued together along $P_+ \cap P_- = \{(z,w) \in P \mid z \in \mathbb{R} \cup \{\infty\}\}$ using the mapping f_q. Two Laguerre planes $\mathcal{L}_q$ and $\mathcal{L}_{q'}$ are isomorphic if and only if $q = q'$.

5.2 The automorphism group. The following facts are taken from [23]; compare also [21, 5.10 Theorem]. Let $\Gamma = \Gamma_q$ denote the automorphism group of $\mathcal{L}_q$. It contains a normal subgroup Π consisting of those automorphisms that fix every parallel class. Π is called the *kernel* of $\mathcal{L}_q$, and it acts on P as follows:

$$(z,w) \mapsto \begin{cases} (z, tw + b_2z^2 + b_1z + b_0), & \text{for } \operatorname{Im}(z) \ge 0, \\ \big(z, tw + f_q^{-1}\big(f_q(b_2)z^2 + f_q(b_1)z + f_q(b_0)\big)\big), & \text{for } \operatorname{Im}(z) \le 0, \\ (\infty, tw + b_2), & \text{for } z = \infty. \end{cases}$$

Here, $b_i \in \mathbb{C}$ is arbitrary, and $t \neq 0$ is real for $q \neq 0$ and complex for the classical plane $\mathcal{L}_0$. The subgroup $\Lambda \leq \Pi$ defined by $t = 1$ is thus isomorphic to $\mathbb{R}^6$ in both cases; its elements will be written as vectors (b_2, b_1, b_0). It is sharply transitive on the set of all circles and is called the *elation group* of the plane. The kernel is a semidirect product $\Pi = \Lambda \rtimes \Omega$, where Ω is defined by $b_1 = b_2 = b_3 = 0$, and is isomorphic to $\mathbb{R}^\times$ or $\mathbb{C}^\times$, respectively. By conjugation, Ω induces on Λ the real or complex scalar multiplications.

Since Λ is sharply transitive on the set of circles, the entire group Γ_q is a semidirect product $\Gamma_q = \Lambda \rtimes \Gamma_{K_0}$, where the right factor is the stabilizer of the circle $K_0 = K_{0,0,0}$. Apart from Ω, this stabilizer contains elements of the following form.

$$(z, w) \mapsto \begin{cases} \left(\frac{az+b}{cz+d}, \frac{w}{(cz+d)^2}\right), & \text{if } \operatorname{Im}(z) \geq 0, cz+d \neq 0, \\ \left(\frac{az+b}{cz+d}, f_q^{-1}\left(\frac{f_q(w)}{(cz+d)^2}\right)\right), & \text{if } \operatorname{Im}(z) \leq 0, cz+d \neq 0, \\ (\infty, c^2 w), & \text{if } cz+d = 0, \\ \left(\frac{a}{c}, \frac{w}{c^2}\right), & \text{if } z = \infty, c \neq 0, \\ (\infty, d^2 w), & \text{if } z = \infty, c = 0, \end{cases}$$

where a, b, c, d are complex or real numbers according to whether $q = 0$ or $q \neq 0$, and satisfy $ad - bc = 1$. These automorphisms form a group Ψ. In order to recognize this group, we look at its action on the first coordinate or, in other words, its action on the set $\mathcal{P}$ of parallel classes. We may regard $\mathcal{P} = \mathbb{C} \cup \{\infty\}$ as the complex projective line, and then Ψ becomes the simple group $\mathrm{PSL}_2\mathbb{C} = \mathrm{PGL}_2\mathbb{C}$ or $\mathrm{PSL}_2\mathbb{R}$, respectively, with its standard action by linear fractional maps. The intersection $\Psi \cap \Omega$ is central in Ψ and therefore reduces to the identity.

Furthermore, the map σ defined by

$$\sigma\colon (z, w) \mapsto \left(-z, f_q^{-1}(iw)\right)$$

for $z \in \mathbb{S}_2 = \mathbb{C} \cup \{\infty\}$, where $-\infty = \infty$, is also an element of Γ_{K_0}. If $q \neq 0$, then $\sigma \notin \Psi$, and this automorphism interchanges the two classical halves of $\mathcal{L}_q$ and normalizes Ψ. The group Φ generated by Ψ together with σ induces $\mathrm{PGL}_2\mathbb{R}$ on $\mathcal{P}$, that is, an extension of $\mathrm{PSL}_2\mathbb{R}$ by $\mathbb{Z}_2$. Note, however, that σ has infinite order; its square is an element of Ω defined by some $t < 0$.

For $q \neq 0$, we have $\Gamma_{K_0} = \Omega \cdot \Phi$; by what we have just seen, this is a nonsplit extension of $\Omega \times \Psi$ by $\mathbb{Z}_2$. For $q = 0$, we define Φ as the (split) extension of Ψ by the involution induced by complex conjugation: $(z, w) \mapsto (\bar{z}, \bar{w})$. Here, we get $\Gamma_{K_0} = \Omega \rtimes \Phi$. To summarize, we have

$$\Gamma_q = (\Lambda \rtimes \Omega) \rtimes \Phi \cong ((\mathbb{R}^6 \rtimes \mathbb{C}^\times) \rtimes \mathrm{PGL}_2\mathbb{C}) \rtimes \mathbb{Z}_2$$

for $q = 0$, and otherwise,

$$\Gamma_q = \Lambda \rtimes (\Omega \cdot \Phi),$$

where

$$\Lambda \cdot \Omega \cong \mathbb{R}^6 \rtimes \mathbb{R}^\times \quad \text{and} \quad (\Omega \cdot \Phi)/\Omega \cong \mathrm{PGL}_2\mathbb{R}.$$

5.3 Choice of an involution. From 5.2 we see that Ω contains an involution ω defined by $t = -1$, and Ψ contains an involution commuting with ω, defined by $(a, b, c, d) = (0, 1, -1, 0)$. None of them is suitable for our purposes, but their product involution τ is; it is given by

$$\tau: P \to P: (z, w) \mapsto \begin{cases} (-1/z, -w/z^2), & \text{for } z \neq 0, \mathrm{Im}(z) \geq 0, \\ \left(-1/z, -f_q^{-1}(f_q(w)/z^2)\right), & \text{for } z \neq 0, \mathrm{Im}(z) \leq 0, \\ (\infty, -w), & \text{for } z = 0, \\ (0, -w), & \text{for } z = \infty. \end{cases}$$

In contrast to the other two, this involution is suited for the construction of a Minkowski plane, because its fixed point set has the right form (two parallel classes), namely

$$\mathrm{Fix}(\tau) = \{(i, w) \mid w \in \mathbb{C}\} \cup \{(-i, w) \mid w \in \mathbb{C}\}.$$

Hence by Theorem 3 we obtain a 4-dimensional Minkowski plane $\mathcal{M}_q = \mathcal{M}(\mathcal{L}_q, \tau)$ from the involution τ. For the action on the circle set one finds that τ maps a circle K_{a_2, a_1, a_0} to the circle $K_{-a_0, a_1, -a_2}$. Thus

$$\mathcal{F} = \{K_{a,b,-a} \mid a, b \in \mathbb{C}\}$$

is the set of circles fixed by τ.

We show next that there are no other ways of obtaining Minkowski planes as brothers of semiclassical Laguerre planes.

5.4 Proposition. *Up to conjugacy in the full automorphism group Γ_q of $\mathcal{L}_q$, the involution τ exhibited in 5.3 is the only involutory automorphism of $\mathcal{L}_q$ fixing two parallel classes pointwise.*

Proof. By 5.2, the group Γ_q is an extension of $\Lambda = \mathbb{R}^6$ by $\Gamma_{K_0} \leq \mathrm{GL}_6\mathbb{R}$, and the action of Γ_q on the circle set is equivalent to the action of this group of affine maps on $\mathbb{R}^6$. Therefore, every involution in Γ_q fixes some circle (take the barycenter of any orbit in $\mathbb{R}^6$). By transitivity, we may assume that the involution fixes K_0. The involution $\omega: (z, w) \mapsto (z, -w)$ belonging to the left factor of $\Gamma_{K_0} = \Omega\Phi$ is clearly unsuitable (for arbitrary q) because it fixes precisely the points of the circle K_0.

¿From Theorem 3, we know that we are looking for involutions that fix exactly two parallel classes (and fix every point on those classes). Such an involution has to preserve the orientation of the 2-sphere $\mathcal{P}$ (the space of parallel classes). For $q = 0$, this excludes the elements of Γ_{K_0} not belonging to $\Omega\Psi$ (i.e., those involving complex conjugation).

In the case $q \neq 0$, the complement $\Omega\Phi \setminus \Omega\Psi$ does not contain any involutions by the results of 5.2. Indeed, in the factor group mod Ω, all such involutions are conjugate to $\sigma\Omega$; however, this coset contains no involution of Γ itself because the square of σ is a non-square element of Ω.

Our search is now reduced to $\Omega \times \Psi$. The second factor Ψ contains just one class of involutions (for arbitrary q). They are not suited by themselves, as we remarked at the beginning of 5.3. The only remaining possibility is to take the product of the unique involution in Ω with the involution just considered. This yields the involution τ of 5.3. $\square$

Finally, we investigate the centralizer of τ in Γ_q for $q \neq 0$. In that case, it will turn out later that this centralizer induces the whole automorphism group of the Minkowski plane $\mathcal{M}_q$.

5.5 Proposition. *For $q \neq 0$, the centralizer $\mathrm{Cs}_\Gamma(\tau)$ of the involution τ (see 5.4) in $\Gamma = \Gamma_q$ is $\mathrm{Cs}_\Gamma(\tau) = \mathrm{Cs}_\Lambda(\tau) \cdot \Omega \cdot \mathrm{Cs}_\Psi(\tau) \cdot \langle\sigma\rangle$. Its factor group $\mathrm{Cs}_\Gamma(\tau)/\langle\tau\rangle$ is a non-split $\mathbb{Z}_2$-extension of a normal subgroup isomorphic to $\mathbb{C}^2 \rtimes \mathbb{C}^\times$, where $\mathbb{C}^\times$ acts on $\mathbb{C}^2$ by the complex scalar multiplications.*

Proof. Clearly, the centralizer is a product of its intersections with each of the factors of the decomposition of Γ_q that we obtained in 5.4. The action of τ on Λ corresponds to the action on circles. It sends the element defined by (b_2, b_1, b_0) to $(-b_0, b_1, -b_2)$ and has a 4-dimensional space of fixed vectors. Direct verification shows that Ω and σ are centralized, and the centralizer of an involution in $\Psi \cong \mathrm{PSL}_2\mathbb{R}$ is well-known to be isomorphic to $\mathrm{SO}_2\mathbb{R}$. It remains to determine how this circle group acts on $\mathbb{R}^4 \leq \Lambda$.

In order to do this, we use the automorphism group Γ_0 of the complex Laguerre plane. The subgroups of Γ_0 and Γ_q that fix the half plane P_+ (compare 5.1) induce the same group on it; hence $\Lambda\Omega\Psi \leq \Gamma_q$ is also a subgroup of Γ_0. As all circle subgroups in $\mathrm{PSL}_2\mathbb{C}$ are equivalent, we may assume that we are dealing with the subgroup $\Xi \leq \Psi_0$ defined by $ad = 1$ and $b = c = 0$ in the notation of 5.2. Its involution is given by $a = i$, and the product of this with ω is $\tau'\colon (z, w) \mapsto (-z, w)$. This involution fixes the elements of Λ given by $(b_2, 0, b_0)$, and Ξ acts on those by $(b_2, 0, b_0) \mapsto a^2(b_2, 0, b_0)$. Since Ω acts in the natural way, it follows that $\Omega\Xi/\langle\tau'\rangle$ acts by complex scalar multiplications. $\square$

We are now ready to prove our first main result:

Theorem 5. *Up to isomorphism, each of the semiclassical Laguerre planes $\mathcal{L}_q$, $0 \leq q < 1$, has a unique brother $\mathcal{M}_q$ that is a 4-dimensional Minkowski plane. The planes $\mathcal{M}_q$ and $\mathcal{M}_{q'}$ are isomorphic if and only if $q = q'$. The plane $\mathcal{M}_0$ is the classical complex Minkowski plane, with a 12-dimensional automorphism group. The other $\mathcal{M}_q$ have 6-dimensional automorphism groups that are non-split extensions of $\mathbb{C}^2 \rtimes \mathbb{C}^\times$ by $\mathbb{Z}_2$. For $q \neq 0$, the automorphism group fixes a point of $\mathcal{M}_q$.*

Proof. The first statement follows from Theorem 3 together with the results of the present section. From the Minkowski plane $\mathcal{M}_q$, $q \neq 0$, its brother $\mathcal{L}_q$ can be recovered by passing to the lifted Lie geometry and then taking the derivation at the unique fixed point of the automorphism group of $\mathcal{M}_q$, whose existence is shown below. For $q = 0$, there is no such fixed point, and the derivation point does not matter. This proves the statement about isomorphisms.

Regarding automorphism groups, observe first that the group $\mathrm{Cs}_{\Gamma_q}(\tau)/\langle\tau\rangle$ described in 5.5 acts faithfully as a subgroup of the automorphism group of $\mathcal{M}_q$ and fixes the special point $\infty = (\infty_1, \infty_2)$ appearing in our construction. Clearly, all automorphisms of the Lie geometry fixing ∞ arise from automorphisms of $\mathcal{L}_q$. According to [18, Corollary 5.38], finally, the automorphism group Σ_q of $\mathcal{M}_q$ is $\mathrm{C}_\tau/\langle\tau\rangle$, where τ is the involution used for the construction of a brother and C_τ denotes its centralizer in the automorphism group of the Lie geometry. We claim that all automorphisms of $\mathcal{M}_q$ fix ∞; then the statements collected here combine to show that Σ_q is in fact given by $\mathrm{Cs}_{\Gamma_q}(\tau)/\langle\tau\rangle$. The existence of a fixed point would follow from [26] if Σ_q were 7-dimensional. If $\dim\Sigma_q \geq 8$, then $\mathcal{M}_q$ is classical by [25], and then $q = 0$. There remains the only possibility that $\dim\Sigma_q = 6$. This implies that ∞ is fixed by Σ_q. Indeed, the stabilizer of ∞ contains the 6-dimensional group Δ inherited from $\mathcal{L}_q$, and all points of $\mathcal{M}_q$ except ∞ have orbits of positive dimension under Δ; thus if the Σ_q-orbit of ∞ is non-trivial, then it has positive dimension, and then $\dim\Sigma_q > \dim\Delta = 6$. $\square$

6. Minkowski planes associated with the semiclassical Laguerre planes

We give a more explicit description of the Minkowski plane $\mathcal{M}_q$, i.e., we now determine the circles, as far as possible, in terms of coordinates. We follow the method of section 4 and consider the trace of $\mathcal{M}_q$ on its derived affine plane at (∞_1, ∞_2).

6.1 The point set of this derived plane is the set $\mathcal{F}$ of circles fixed by τ in $\mathcal{L}_q$. As explained in section 4, the fixed circles are identified with their pairs of intersection points with the two distinguished parallel classes π_1 and π_2. More precisely, if the intersection points are $(i, c') \in \pi_1$ and $(-i, d') \in \pi_2$, then the fixed circle is represented by the pair (c', d'). Thus the affine plane has point set $\pi_1 \times \pi_2 \cong \mathbb{C}^2$ and we have the canonical parallelisms. The last fact is our reason for choosing this particular coordinate system.

According to 5.3 a fixed circle has the form $K_{a,b,-a}$. It intersects π_1 and π_2 in $(i, -2a+bi)$ and $\big(-i, f_q^{-1}(-2f_q(a) - f_q(b)i)\big)$, respectively. Conversely, for given $c', d' \in \mathbb{C}$ there is precisely one circle of the form $K_{a,b,-a}$ through (i, c') and $(-i, d')$, namely for $a = \frac{1}{4}f_q(c'-d') - \frac{1}{2}c$ and $b = \frac{1}{2}if_q(d'-c')$; compare section 4. We used primes because we want to change coordinates by

$$(c, d) = \big(c', f_q(d')\big).$$

We can now translate the coordinates (a, b) of a fixed circle to (c, d) and back by

$$(c, d) = \big(-2a + bi, -2f_q(a) - f_q(b)i\big)$$

and

$$\begin{aligned}(a, b) &= \Big(\frac{1}{4}\big(f_q(c) - 2c - d\big), \frac{1}{2}i\big(d - f_q(c)\big)\Big)\\ &= \Big(-\frac{1}{4}(d + c - q\bar{c}), \frac{1}{2}i(d - c - q\bar{c})\Big).\end{aligned}$$

6.2 The group of translations. According to Theorem 5, the centralizer $\mathrm{Cs}_\Lambda(\tau)$ acts on $\mathcal{M}_q$ by automorphisms. A typical element of this group sends the fixed circle

(= point of $\mathcal{M}_q$) defined by $(a, b) \in \mathbb{C} \times \mathbb{C}$ to the one defined by $(a + a_0, b + b_0)$. In the (c, d)-coordinate system, the same map acts by $(c, d) \mapsto \big(c - 2a_0 + b_0 i, d - 2f_q(a_0) - f_q(b_0)i\big) = (c + c_0, d + d_0)$. Thus in either coordinate system, we may use arbitrary translations in order to simplify the equations of circles. We shall use this fact mainly in 6.4.

6.3 Lines of the affine plane are precisely the parallel classes and the circles of $\mathcal{M}_q$ that pass through (∞_1, ∞_2). The latter have the form $(z, w)^\perp$ for points (z, w) of the Laguerre plane not on $\pi_1 \cup \pi_2$, that is, for $z \notin \{i, -i\}$. A point (c, d) belongs to this line if the fixed circle $K_{a,b,-a}$ with coordinates (c, d) passes through (z, w) in the Laguerre plane. We distingush two cases.

Case 1. Let $z \in \mathbb{C}$, $\mathrm{Im}(z) \geq 0$, $z \neq i$. Circles $K_{a,b,-a}$ in $\mathcal{F}$ that pass through (z, w) satisfy

$$\begin{aligned} w &= a(z^2 - 1) + bz \\ &= -\frac{1}{4}(d + c - q\bar{c})(z^2 - 1) + \frac{1}{2}i(d - c - q\bar{c})z \\ &= -\frac{1}{4}(z + i)^2 c + \frac{1}{4}q(z - i)^2 \bar{c} - \frac{1}{4}(z - i)^2 d. \end{aligned}$$

Therefore

$$d = -\left(\frac{z + i}{z - i}\right)^2 c + q\bar{c} - \frac{4w}{(z - i)^2}.$$

The transformation $z \mapsto -(\frac{z+i}{z-i})^2$ takes the region $\mathrm{Im}(z) \geq 0$, $z \neq i$ onto $\{m \in \mathbb{C} \mid \|m\| \geq 1\}$. Hence we obtain *lines of the first type,* defined by the equations

$$d = mc - q\bar{c} + t$$

for all $m, t \in \mathbb{C}$, $\|m\| \geq 1$.

Case 2. For $z \in \mathbb{C}$, $\mathrm{Im}(z) \leq 0$, $z \neq -i$ we similarly obtain

$$\begin{aligned} f_q(w) &= f_q(a)(z^2 - 1) + f_q(b)z \\ &= -\frac{1}{4}f_q(d + c - q\bar{c})(z^2 - 1) + \frac{1}{2}f_q\big(i(d - c - q\bar{c})\big)z \\ &= -\frac{1}{4}(1 - q^2)(z + i)^2 c - \frac{1}{4}(z - i)^2 d - \frac{1}{4}q(z + i)^2 \bar{d}. \end{aligned}$$

Therefore

$$c = -\frac{1}{1 - q^2}\left(\left(\frac{z - i}{z + i}\right)^2 d + q\bar{d} + \frac{4f_q(w)}{(z + i)^2}\right).$$

The transformation $z \mapsto -(\frac{z-i}{z+i})^2$ takes the region $\mathrm{Im}(z) \leq 0$, $z \neq -i$ onto $\{m \in \mathbb{C} \mid \|m\| \geq 1\}$. Hence we obtain *lines of the second type,* defined by the equations

$$c = \frac{m}{1 - q^2}d - \frac{q}{1 - q^2}\bar{d} + t$$

for all $m, t \in \mathbb{C}$, $\|m\| \geq 1$. In Theorem 6, we shall use these equations in their present form, but they can also be solved for d. One finds

$$d = \frac{1-q^2}{\|m\|^2 - q^2}(\bar{m}c + q\bar{c}) + t'$$

for some $t' \in \mathbb{C}$. By 6.2, every complex number t' occurs for a given m.

6.4 Circles of $\mathcal{M}_q$ in the narrow sense (i.e., those not passing through (∞_1, ∞_2)) are of the form $C^\perp \cap \mathcal{F}$ for circles C of $\mathcal{L}_q$ not belonging to $\mathcal{F}$, that is, $C = K_{\alpha,\beta,\gamma}$ with $\alpha, \beta, \gamma \in \mathbb{C}$, $\alpha + \gamma \neq 0$. Using translations, we may simplify this to $\alpha = \beta = 0$, see 6.2. We have to determine the circles $K_{a,b,-a} \in \mathcal{F}$ that touch $K_{0,0,\gamma}$, where $\gamma \neq 0$.

Suppose first that the first coordinate $z \in \mathbb{C}$ of the touching point satisfies $\mathrm{Im}(z) \geq 0$. We must have $z \neq i$, because a fixed circle and a non-fixed circle cannot touch at $(i, w) \in \pi_1$. Since touching is equivalent to intersection with equal derivatives, see [23, 4.2] or [24, 5.10], we obtain the equations

$$\begin{aligned}\gamma &= a(z^2-1) + bz,\\ 0 &= 2az + b,\end{aligned}$$

with the solution

$$\begin{aligned}a &= -\frac{\gamma}{z^2+1},\\ b &= \frac{2\gamma z}{z^2+1}.\end{aligned}$$

We eliminate the parameter z in these parametric equations for half the circle. We find $z = -\frac{b}{2a}$ and substitute this expression in the equation for a. We obtain

$$b^2 + (2a+\gamma)^2 = \gamma^2.$$

In our (c,d)-coordinate system this equation becomes

$$-(d - c - q\bar{c})^2 + (d + c - q\bar{c} - 2\gamma)^2 = 4\gamma^2.$$

If this holds with $c = \gamma$, then it follows that $\gamma = 0$, contrary to our assumption. Writing the parentheses as $(d+x)$ and $(d+y)$, respectively, we obtain the equation $(y-x)(2d+y+x) = 4\gamma^2$, and $y - x = 2(c-\gamma) \neq 0$. Hence the following equation describes one half of the circle:

$$d = \frac{\gamma^2}{c-\gamma} + q\bar{c} + \gamma.$$

We claim that here c ranges over all complex numbers satisfying $c \notin \{0, \gamma\}$ and $\mathrm{Re}\,\frac{\gamma}{c} \geq \frac{1}{2}$. This is obtained in several steps. First, we translate the previous conditions $\mathrm{Im}\, z \geq 0$ and $z \neq i$ to $\mathrm{Im}\,\frac{b}{a} \leq 0$ and $b + 2ai \neq 0$. The formulae in 6.1

for coordinate changes together with the above identity give $\operatorname{Im}\frac{b}{a} = -2\operatorname{Re}(\frac{2\gamma}{c} - 1)$, which yields our claim.

There remains the case where the first coordinate $z \in \mathbb{C}$ of the touching point satisfies $\operatorname{Im}(z) \leq 0$ and $z \neq -i$. In this case we similarly obtain

$$\begin{aligned} f_q(\gamma) &= f_q(a)(z^2-1) + f_q(b)z, \\ 0 &= 2f_q(a)z + f_q(b). \end{aligned}$$

Therefore,

$$\begin{aligned} f_q(a) &= -\frac{f_q(\gamma)}{z^2+1}, \\ f_q(b) &= \frac{2f_q(\gamma)z}{z^2+1}. \end{aligned}$$

Eliminating the parameter $z = -\frac{f_q(b)}{2f_q(a)}$ we now find

$$f_q(b)^2 + f_q(2a+\gamma)^2 = f_q(\gamma)^2.$$

In (c, d)-coordinates after some manipulations similar to the previous ones this equation becomes

$$(1-q^2)c = \frac{f_q(\gamma)^2}{d - f_q(\gamma)} - q\bar{d} + f_q(\gamma).$$

This equation describes the second half of our circle; here, d ranges over all complex numbers $d \notin \{0, f_q(\gamma)\}$ satisfying $\operatorname{Re}\frac{f_q(\gamma)}{d} \geq \frac{1}{2}$.

In order to simplify the equations for both halves of the circle, we apply the translation

$$(c, d) \mapsto (c - \gamma, d - f_q(\gamma)).$$

This transforms the two equations respectively to

$$d = \frac{\gamma^2}{c} + q\bar{c},$$

where $c \in \mathbb{C} \setminus \{0, -\gamma\}$ and $\operatorname{Re}\frac{\gamma}{\gamma+c} \geq \frac{1}{2}$, and to

$$(1-q^2)c = \frac{f_q(\gamma)^2}{d} - q\bar{d},$$

where $d \in \mathbb{C} \setminus \{0, -f_q(\gamma)\}$ and $\operatorname{Re}\frac{f_q(\gamma)}{d+f_q(\gamma)} \geq \frac{1}{2}$.

We wish to express the inequalities involving the real parts in a more convenient way. We use the fact that a complex number u satisfies $\operatorname{Re} u \geq \frac{1}{2}$ if and only if $\|\frac{1}{u} - 1\| \leq 1$. We obtain the conditions $\|c\| \leq \|\gamma\|$ and $\|d\| \leq \|f_q(\gamma)\|$, respectively.

Finally, we check that the two halves of our circle fit together nicely. Indeed, if $\|c\| = \|\gamma\| \neq 0$ and $d = \gamma^2 c^{-1} + q\bar{c}$, then we obtain readily that also $\|d\| = \|f_q(\gamma)\|$ and $(1-q^2)cd = f_q(\gamma)^2 - q\|d\|^2$. The converse statement is also true. In summary, we have obtained the following.

Theorem 6. *The 4-dimensional Minkowski plane $\mathcal{M}_q$, $0 \le q < 1$, constructed as a brother of the Laguerre plane $\mathcal{L}_q$ with respect to the involution τ may be described as follows. By deleting the points of the two distinguished parallel classes one obtains an affine translation plane $\mathcal{A}_q$ with point set $\mathbb{C}^2$. Its lines are all translates of the coordinate axes and of the following linear subspaces:*

$$\{(c, mc - q\bar{c}) \mid c \in \mathbb{C}\},$$

$$\left\{ \left(\frac{m}{1-q^2}d - \frac{q}{1-q^2}\bar{d}, d\right) \,\middle|\, d \in \mathbb{C} \right\},$$

for all $m \in \mathbb{C}$ satisfying $\|m\| \ge 1$.

The lines of $\mathcal{A}_q$ are the circles of $\mathcal{M}_q$ passing through its distinguished point together with the parallel classes. The remaining circles are described as unions of a function graph with the graph of a function in the reverse direction. Up to a translation of $\mathbb{C}^2$, the parts have the form

$$\left\{ \left(c, \frac{\gamma^2}{c} + q\bar{c}\right) \,\middle|\, c \in \mathbb{C} \setminus \{0, -\gamma\}, \|c\| \le \|\gamma\| \right\},$$

and

$$\left\{ \left((1-q^2)^{-1}\left(\frac{f_q(\gamma)^2}{d} - q\bar{d}\right), d\right) \,\middle|\, d \in \mathbb{C} \setminus \{0, -f_q(\gamma)\}, \|d\| \le \|f_q(\gamma)\| \right\},$$

where $\gamma \in \mathbb{C} \setminus \{0\}$. $\square$

It seems impossible to explicitly verify the axioms of a Minkowski plane given the above description of the circles. However, note that for $q = 0$ we have $f_0 =$ id and the circles reduce to the familiar forms for Euclidean lines and Euclidean hyperbolae.

6.5 Automorphisms. Direct verification shows that the following maps are automorphisms of $\mathcal{M}_q$:

$$(c, d) \mapsto (\bar{z}c + u, zd + v),$$

where $z, u, v \in \mathbb{C}, z \ne 0$. They form a group isomorphic to $\mathbb{C}^2 \cdot \mathbb{C}^\times$, which must be the identity component of the automorphism group Σ_q of $\mathcal{M}_q$, compare Theorem 5. It remains to describe the action of the automorphism σ introduced in 5.2. There, σ is given as a map of the point set P. Using the original coordinates for fixed circles of $\mathcal{L}_q$ (i.e., (a, b) for $K_{(a,b,-a)}$), we can express its effect on the point set of $\mathcal{M}_q$ by the map $\sigma: \mathbb{C}^2 \to \mathbb{C}^2: (a, b) \mapsto (f_q^{-1}(ia), -f_q^{-1}(ib))$. By the coordinate change $(a, b) \mapsto (c, d)$ introduced in 6.1, this is transformed into

$$\sigma: (c, d) \mapsto \left(\frac{id}{1-q^2}, ic\right).$$

This automorphism interchanges the two parallelisms of $\mathcal{M}_q$, and σ^2 is multiplication by $-\frac{1}{1-q^2}$.

6.6 Derived affine plane. We have a closer look at the description of $\mathcal{A}_q$ in (a, b)-coordinates that we obtained as the first step in 6.3. Given $z, w \in \mathbb{C}$, $z \notin \{i, -i\}$, we obtained these equations for lines: $w = a(z^2-1) + bz$ for $\operatorname{Im} z \geq 0$ and $f_q(w) = f_q(a)(z^2-1) + f_q(b)z$ for $\operatorname{Im} z \leq 0$. This shows immediately that $\mathcal{A}_q$ is a translation plane. The defining spread (consisting of the lines that pass through the origin) is obtained if we choose $w = 0$. We observe in passing that the spread is made up of two halves isomorphic to the complex spread. Taking $z \in \{1, -1, 0\}$ gives the coordinate axes. We shall not exclude the values $z \in \{i, -i\}$ and obtain a recognizable spread. This must be the correct one, because any two lines of a spread are determined by the remaining ones.

Let $m = \frac{-z}{z^2-1}$, $z \neq \pm 1$. Then the imaginary parts of z and m have the same sign, and the equations for lines passing through the origin may be rewritten as $a = mb$ for $\operatorname{Im} m \geq 0$ and $f_q(a) = mf_q(b)$ for $\operatorname{Im} m \leq 0$. Using the inversion formula for f_q, see 5.1, the second equation may be expressed as $a = nb + \varphi(n)\bar{b}$, where $\varphi(x+iy) = \frac{2q}{1+q^2} iy$. By comparison with the information given in [13, Theorem 73.13] (including the subsequent Note (c)), we may identify our spread as follows.

Theorem 7. *The derived affine plane $\mathcal{A}_q$ of the Minkowski plane $\mathcal{M}_q$ obtained by derivation at the distinguished point (∞_1, ∞_2) is isomorphic to one of the 'semiclassical' translation planes admitting an irreducible action of* $\mathrm{SL}_2\mathbb{R}$. *The parameter defining the isomorphism type as in* [13, 73.13] *is* $\left(\frac{1+q}{1-q}\right)^2 \geq 1$. *The automorphism group of $\mathcal{A}_q$ is 8-dimensional if $q \neq 0$.* □

Note that the automorphism group of $\mathcal{M}_q$ is only 6-dimensional. Hence, $\mathcal{A}_q$ has many automorphisms that change $\mathcal{M}_q$; in particular, on $\mathcal{A}_q$ there lives a 2-dimensional family of copies of $\mathcal{M}_q$.

7. Alternative approach via semiclassical pseudo-ovals

The approach presented in this section is based on the theory of pseudo-ovals developed in [11]. The essential idea is that one constructs the semiclassical Laguerre plane $\mathcal{L}_q$ and its Lie geometry directly from the elation group $\Lambda \cong \mathbb{R}^6$ (see 5.2). Both structures are encoded in the family of all stabilizers Λ_p, where p is a point of $\mathcal{L}_q$. This is a family of 4-dimensional subspaces of $\mathbb{R}^6$ which is compact in the Grassmann topology and has very peculiar geometric properties. Namely, the intersections of all these subspaces with any given one of them almost form a compact spread; only one element of the spread is missing. A family $\mathcal{T}$ of subspaces with these properties has been called a *dual pseudo-oval* in [11]. The missing spread elements together form the *pseudo-oval* $\mathcal{O}$ associated with the dual pseudo-oval $\mathcal{T}$. By the duality theorem [11, 7.1], this is really a pseudo-oval in the sense that a spread is induced on $\mathbb{R}^6/o$ for each $o \in \mathcal{O}$; at this point, the compactness is used. The elements of $\mathcal{T}$ are sometimes called the *tangents* of $\mathcal{O}$. As a consequence of the duality theorem, $\mathcal{T}$ and $\mathcal{O}$ together form a *Kantor family* or *4-gonal family* in the sense of [2].

The main advantage of this way of looking at the semiclassical Laguerre plane $\mathcal{L}_q$ is that its automorphism group Γ_q appears as a group of affine transformations of the elation group Λ (which may be identified with the set of circles). The identity component of the automorphism group Γ_q acts as a subgroup of the classical group Γ_0. By contrast, in 5.2 we had to write down each automorphism using case distinctions.

The elements of a dual pseudo-oval may be written down conveniently as kernels of a parametrized family of matrices. We shall use this in order to obtain an alternative description of the brother Minkowski plane $\mathcal{M}_q$ of $\mathcal{L}_q$ in terms of coordinates. We shall obtain a variation of the description in terms of (a,b)-coordinates given in section 6. Thus we will not get the canonical parallelism, and the circles will be parametrized rather than composed of function graphs. However, the lines will be obtained in a very explicit form that practically coincides with the standard description of the semiclassical translation planes.

Notation. Matrices act from the right on row vectors. A (6×2)-matrix will often be written in block form

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix},$$

the blocks being of size 2×2. The unit and zero (2×2)-matrices will be denoted by $\mathbf{1}$ and $\mathbf{0}$, respectively.

7.1 The classical dual pseudo-oval is an oval, in fact, a conic in the dual of the complex projective plane. Consider the complex conic whose points have homogeneous coordinates $[0,0,1]$ and $[1,-2z,z^2]$, $z \in \mathbb{C}$. It is defined by the quadratic form $g(u,v,w) = 2uw - \frac{1}{2}v^2$, and hence its tangent at the point with parameter z is the kernel of the linear form $(u,v,w) \mapsto z^2u + zv + w$ on $\mathbb{C}^3$. The tangent at the remaining point has equation $u = 0$. If we consider these kernels as 4-dimensional real subspaces of $\mathbb{R}^6$, then we obtain the classical dual pseudo-oval $\mathcal{T}_0$. This yields the following (6×2)-matrices whose kernels are the elements of $\mathcal{T}_0$:

$$D_0(\infty) = \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \quad \text{and} \quad D_0(z) = \begin{pmatrix} B_0(z)^2 \\ B_0(z) \\ \mathbf{1} \end{pmatrix},$$

where $z = x + iy \in \mathbb{C}$ and

$$B_0(z) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}.$$

7.2 Semiclassical dual pseudo-ovals. In order to obtain a dual pseudo-oval $\mathcal{T}_q$ by distorting the classical example, we introduce a function of a real variable by

$$\mu(y) = \begin{cases} \frac{1-q}{1+q}, & \text{if } y < 0, \\ 1, & \text{if } y \geq 0, \end{cases}$$

and we form the matrices

$$B(z) = B(x+iy) = \begin{pmatrix} x & y \\ -\mu(y)^2 y & x \end{pmatrix}.$$

Now the matrices defining $\mathcal{T}_q$ are $D(\infty) = D_0(\infty)$ (the same as in 7.1) and

$$D(z) = \begin{pmatrix} B(z)^2 \\ B(z) \\ \mathbf{1} \end{pmatrix}$$

for all $z \in \mathbb{C}$. These matrices are not quite the same as in [11], because in that paper the coordinate transformation $(x_1, ..., x_6) \mapsto (-x_1, -x_2, -x_4, -x_3, x_5, x_6)$ of $\mathbb{R}^6$ was used in order to match the matrices with those given in [24]. Moreover, due to our use of row vectors, our matrices here had to be transposed.

We shall also need the pseudo-oval $\mathcal{O}_q$ associated with $\mathcal{T}_q$, which consists of the 2-dimensional subspaces of $\mathbb{R}^6$ that have to be added to the intersections of one tangent with all other ones in order to obtain a spread. According to [11, 4.7], these subspaces are obtained by intersecting, for each z, the kernels of $D(z)$ and of the partial derivatives

$$\frac{\partial}{\partial x} D(z) = \begin{pmatrix} 2B(z) \\ \mathbf{1} \\ \mathbf{0} \end{pmatrix}$$

and $\frac{\partial}{\partial y} D(z)$. The latter is equal to $\frac{\partial}{\partial x} D(z) \cdot A(z)$, where $A(z)$ is the invertible matrix

$$A(z) = \begin{pmatrix} 0 & 1 \\ -\mu(y)^2 & 0 \end{pmatrix}.$$

The intersection is generated by the rows of the (2×6)-matrix

$$D^*(z) = \left(\mathbf{1}, -2B(z), B(z)^2\right).$$

(The partials exist only for $y \neq 0$, but the result remains true for real z by continuity.) Moreover, we have

$$D^*(\infty) = (\mathbf{0}, \mathbf{0}, \mathbf{1}).$$

Note that for $q = 0$, this reproduces the conic containing the points $[1, -2z, z^2]$ for $z \in \mathbb{C}$.

7.3 The connection to semiclassical Laguerre planes and quadrangles. Consider an element $\lambda = (b_2, b_1, b_0) \in \Lambda = \mathbb{R}^6$. Using the matrices $D(z)$ that parametrize the semiclassical dual pseudo-oval, we may write the action of λ on the points of the semiclassical Laguerre plane $\mathcal{L}_q$ as

$$\lambda(\hat{z}, w) = \left(\hat{z}, w + \lambda D(z)\right),$$

where $\widehat{x + iy} = x + i\mu(y)y$ and $\widehat{\infty} = \infty$. This claim is best verified by testing it on the standard basis vectors (b_2, b_1, b_0) with $b_j \in \{0, 1, i\}$; this suffices by $\mathbb{R}$-linearity.

Thus we see that the elements of $\mathcal{T}_q$ are precisely the stabilizers of Λ acting on the points of $\mathcal{L}_q$. Conversely, it is easy to reconstruct the Laguerre plane from the pseudo-oval, because Λ is sharply transitive on circles. According to [11, 6.1] we obtain the following description of $\mathcal{L}_q$. Points are the cosets of the elements $t \in \mathcal{T}_q$

in Λ, circles are the elements of Λ, incidence is the natural one, and parallelism of points is the natural parallelism of cosets, that is, two points $t+\lambda$, $t'+\lambda'$ are parallel if and only if $t=t'$.

When do two circles touch? Suppose the circles are represented by $\alpha, \beta \in \Lambda$. We ask under which conditions there is exactly one point $t+\lambda$, $t \in \mathcal{T}_q$, containing both of them. We may assume that $\lambda = \alpha$. Moreover, up to a translation we have $\alpha = 0$. Then our question becomes: which $\beta \neq 0$ are contained in exactly one tangent t? The answer is clear; β has to belong to the element $p_t \in \mathcal{O}_q$ determined by t. In full generality this means that the touching pencils are precisely the cosets of the elements of $\mathcal{O}_q$. If $0 \neq \alpha - \beta \in p_t$, then these two circles touch at the point $t+\alpha$.

Now we can easily describe the quadrangle $\mathcal{Q}_q$ which is obtained as the Lie geometry of $\mathcal{L}_q$. After deleting the special point ∞, the remaining points are the elements of Λ and all cosets of all elements of $\mathcal{T}_q$, and the lines are all cosets of all elements of $\mathcal{O}_q$ and the sets $\{t+\lambda \mid \lambda \in \Lambda\}$ for all $t \in \mathcal{T}_q$. Incidence is inclusion, that is, the line $p_t + \lambda$ is incident with the points represented by its own elements and with the point $t+\lambda$, and a line of the second kind is incident with its own elements (and with ∞). Note that the symbol ∞ is also used for the special element in the one-point compactification of $\mathbb{C}$; this should not lead to confusion. We remark that our construction of $\mathcal{Q}_q$, which is based on [11], is also a special case of [2, Theorem 2.2].

7.4 Automorphism group. By an automorphism of a dual pseudo-oval $\mathcal{T}$ we mean an automorphism of the vector space $\mathbb{R}^6$ that permutes the elements of $\mathcal{T}$. From 5.2 we know that the automorphism group Γ_q of $\mathcal{L}_q$ is a semidirect product $\Gamma_q = \Lambda \rtimes \Gamma_K$ of Λ with the stabilizer of any circle K. Therefore, Γ_K acts effectively on Λ by conjugation. From 7.3 it follows that Γ_K induces automorphisms of $\mathcal{T}_q$, and that all automorphisms of $\mathcal{T}_q$ arise in this way.

In the classical case $q=0$, we see that the automorphism group of the pseudo-oval $\mathcal{T}_0$ is generated by complex conjugation together with the similarity transformations with respect to the quadratic form $g(u,v,w) = 2uw - \frac{1}{2}v^2$ defining the pseudo-oval. The semisimple part of this group is a group $\mathrm{SO}_3(\mathbb{C},1)$ that acts on the parameter z by linear fractional maps.

In the proof of 5.5 we have seen that the identity component of the automorphism group Γ_q of $\mathcal{L}_q$ is a subgroup of Γ_0. This carries over to the automorphism groups of the corresponding pseudo-ovals. It follows that the automorphism group of $\mathcal{T}_q$ contains a subgroup $\Psi = \mathrm{SO}_3(\mathbb{R},1) \leq \mathrm{SO}_3(\mathbb{C},1)$ that acts on z by linear fractional maps with real coefficients. Moreover, the group Ω of multiplications by real scalars is contained in the automorphism group of $\mathcal{T}_q$. These two groups leave the classical halves of $\mathcal{T}_q$ invariant. Finally, we have the automorphism σ, whose inverse acts on Λ by

$$\sigma^{-1}(u,v,w) = \big(-if_q(u), if_q(v), -if_q(w)\big)$$

and interchanges the classical halves. By 5.2, the automorphism group of $\mathcal{T}_q$ is generated by the subgroups enumerated here.

7.5 The involution. We recall from the proof of 5.5 that the involution τ defining the brother $\mathcal{M}_q$ of $\mathcal{L}_q$ acts on Λ by $(u,v,w) \mapsto (-w,v,-u)$. It is not hard to compute the action of this map on $\mathcal{T}_q$; it sends the kernel of $D(z)$ to that of $D(w)$, where w satisfies $B(z)B(w) = -1$. Thus the image is the kernel of $D(-\bar{z}\|\hat{z}\|^{-2})$. (If z has positive imaginary part, then this coincides with $D(-z^{-1})$.) Later we shall only need to know the fixed tangents; hence we prefer to determine them by exploiting the connection with the Laguerre plane, see 7.3. We obtain that the fixed elements of τ in $\mathcal{T}_q$ are precisely the stabilizers of the fixed points of τ in $\mathcal{L}_q$, that is, the stabilizers of (i, w) and of $(-i, w)$, where $w \in \mathbb{C}$ is arbitrary. These stabilizers are the kernels of $D(z)$ for z satisfying $\hat{z} = \pm i$, that is, the kernels of $D(i)$ and $D(-i/\mu)$. (More precisely, we should write $\mu(-1)$ instead of μ at this point.) Direct computation shows that τ also fixes every coset of these spaces. Moreover, τ fixes the corresponding elements of $\mathcal{O}_q$, generated by the rows of $D^*(i)$ and $D^*(-i/\mu)$, respectively (but not their cosets).

7.6 The Minkowski plane $\mathcal{M}_q$. We want to apply Theorem 1 to the involution τ on the quadrangle $\mathcal{Q}_q$, using the description of $\mathcal{Q}_q$ given in 7.3. For the sake of simplicity, we shall write

$$t(z) := \ker D(z),$$

and we use $p(z)$ to denote the corresponding element of $\mathcal{O}_q$, which is generated by the rows of $D^*(z)$. The points of $\mathcal{M}_q$ are by definition the points fixed by τ in $\mathcal{Q}_q$, that is, the elements fixed by τ in Λ, the point ∞, and the cosets of $t(i)$ and $t(-i/\mu)$. Those cosets will be regarded as points at infinity, i.e., we shall represent $\mathcal{M}_q$ in the derived affine plane $\mathcal{A}_q$ taken at ∞, whose point set S consists of the fixed elements in Λ:

$$S = \{(u, v, -u) \mid u, v \in \mathbb{R}^2\} \leq \Lambda.$$

The parallel classes of $\mathcal{M}_q$ are the lines fixed by τ in $\mathcal{Q}_q$. Thus the two parallel classes containing ∞ consist of the cosets of $t(i)$ or $t(-i/\mu)$, respectively. All other parallel classes are cosets $p(i) + s$ or $p(-i/\mu) + s$ contained in S. Strictly speaking, these classes also contain points at infinity, namely the points $t(i)+s$ or $t(-i/\mu)+s$, respectively.

A circle of $\mathcal{M}_q$ consists of all fixed points in $\mathcal{Q}_q$ that are collinear with a given non-fixed point r. Two cases are possible. If r is collinear to ∞, then we obtain a circle passing through ∞, i.e., a line of $\mathcal{A}_q$. If r is not collinear to ∞, then we obtain a proper circle. Points collinear to ∞ have the form $r = t(z) + \lambda$, where $\lambda \in \Lambda$ and $z \in \mathbb{C} \cup \{\infty\}$. If r is not fixed by τ, then $z \notin \{i, -i/\mu\}$. In this case, we obtain the affine line

$$S \cap \big(t(z) + \lambda\big).$$

Indeed, a point $\alpha \in S$ can be joined to r if and only if it is an element of r (in which case the joining line is the coset $p(z) + \alpha$). We emphasize the surprising fact that S is another subspace (besides the elements of $\mathcal{T}_q$) on which the pseudo-oval $\mathcal{T}_q$ induces a translation plane via intersection.

A circle not passing through ∞ is obtained if $r = \lambda \in \Lambda \setminus S$. Lines of $\mathcal{Q}_q$ passing through such a point r are necessarily of the form $p(z)+\lambda$ for some $z \in \mathbb{C}\cup\{\infty\}$. All elements of S belonging to one of these lines constitute the affine part of the circle defined by λ. In addition, this circle contains the elements $t(i) + \lambda$ and $t(-i/\mu) + \lambda$ at infinity.

The description of $\mathcal{M}_q$ so obtained will be made more explicit in the remaining subsections.

7.7 The derived affine translation plane. We identify the element $(u, v, -u) \in S$ with $(u, v) \in \mathbb{R}^2 \times \mathbb{R}^2$. The translation plane $\mathcal{A}_q$ is determined by the spread formed by the lines through the origin. Commonly the elements of a spread in $\mathbb{R}^2 \times \mathbb{R}^2$ are represented by their Grassmann coordinates, compare [13, 64.8]; to do so, one specifies a set of (2×2)-matrices M such that the spread consists of the right factor $\{0\} \times \mathbb{R}^2$ and the subspaces $\{(u, uM) \mid u \in \mathbb{R}^2\}$.

In our case, the lines through the origin are $S \cap t(z)$, where $z \notin \{i, -i/\mu\}$, and $p(i), p(-i/\mu)$. The latter two are the parallel classes of the origin. Under the above identification, $p(z)$ corresponds to the space generated by the rows of $(\mathbf{1}, -2B(z))$. Thus we have two lines, generated by the rows of the matrices

$$\begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 2 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 2\mu^{-1} \\ 0 & 1 & -2\mu & 0 \end{pmatrix},$$

and the corresponding spread matrices are

$$\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 2\mu^{-1} \\ -2\mu & 0 \end{pmatrix}.$$

The line defined by $t(\infty)$ is the right factor of $\mathbb{R}^2 \times \mathbb{R}^2$. Finally, consider $z \in \mathbb{C} \setminus \{i, -i/\mu\}$. An element $(u, v, -u) \in S$ belongs to the kernel of $D(z)$ if and only if

$$v = \big(u - uB(z)^2\big)B(z)^{-1}.$$

Thus we have the spread matrices $B(z)^{-1} - B(z)$. For $z = 0$ we obtain the right factor once more, and for $z \neq 0$ we have

$$B(z)^{-1} = \alpha \begin{pmatrix} x & -y \\ \mu(y)^2 y & x \end{pmatrix}, \quad \text{where} \quad \alpha = \frac{1}{\|\hat{z}\|^2}.$$

Thus the spread matrix is

$$\begin{pmatrix} (\alpha - 1)x & -(\alpha + 1)y \\ (\alpha + 1)\mu(y)^2 y & (\alpha - 1)x \end{pmatrix}.$$

This is a spread matrix of the translation plane with reducible $\mathrm{SL}_2\mathbb{R}$-action and parameter μ^{-2} as presented in [13, 73.13]. (Observe that matrices act from the left in [13], hence we have to compare the transposes of our matrices to the matrices given there.) The two lines obtained above from parallel classes also belong to this spread. This confirms our result obtained in Theorem 7.

7.8 Proper circles. Consider an element $\lambda = (a, b, c) \in \Lambda$ that is not fixed by τ, i.e., $c \neq -a$. The affine part of the circle defined by λ consists of all (u, v) such that $(u, v, -u) \in p(z) + \lambda$ for some $z \in \mathbb{C} \cup \{\infty\}$. For $z = \infty$, where $D^*(z) = (\mathbf{0}, \mathbf{0}, \mathbf{1})$, the only solution is $(u, v) = (a, b)$. For $z \neq \infty$, the space $p(z)$ is generated by the rows of the matrix $D^*(z) = \big(\mathbf{1}, -2B(z), B(z)^2\big)$. Therefore, the elements of $p(z)$ have the form $\big(h + a, -2hB(z) + b, hB(z)^2 + c\big)$, $h \in \mathbb{R}^2$, and the circle consists of (a, b) and all (u, v) such that there exist $z \in \mathbb{C}$ and $h \in \mathbb{R}^2$ satisfying the equation

$$\big(h + a, -2hB(z) + b, hB(z)^2 + c\big) = (u, v, -u).$$

For a given z, this amounts to a system of equations for h. If $B(z)^2 + \mathbf{1}$ is invertible, then the system has a unique solution; using $d = -(a + c) \neq 0$ we obtain $h = d\big(B(z)^2 + \mathbf{1}\big)^{-1}$, hence

$$\begin{aligned} u &= d\big(B(z)^2 + \mathbf{1}\big)^{-1} + a, \\ v &= -2d\big(B(z)^2 + \mathbf{1}\big)^{-1} B(z) + b. \end{aligned}$$

Observe how λ enters into the result: (a, b) represents a translation of the whole circle, and $d = -(a+c)$ is an arbitrary nonzero vector. It remains to see if $B(z)^2 + \mathbf{1}$ is invertible and to write down the matrices appearing in the equations for u and v. We have

$$\delta_z := \det\big(B(z)^2 + \mathbf{1}\big) = \big(x^2 - \mu(y)^2 y^2 + 1\big)^2 + \big(2\mu(y)xy\big)^2 \geq 0,$$

and the determinant vanishes if and only if $x = 0$ and $\mu(y)^2 y^2 = 1$, i.e., if and only if $z \in \{i, -i/\mu\}$. In that case, $B(z)^2 + \mathbf{1} = \mathbf{0}$, and there is no solution h.

In summary, up to translation in $S = \mathbb{R}^4$ every proper circle is obtained from $\lambda = (0, 0, -d)$ for some $d \in \mathbb{R}^2 \setminus \{0\}$, and this circle contains the origin 0 and the elements

$$\delta_z^{-1} d \begin{pmatrix} x^2 - \mu^2 y^2 + 1 & -2xy & -2(\gamma_z + 1)x & 2(\gamma_z - 1)y \\ 2\mu xy & x^2 - \mu^2 y^2 + 1 & -2\mu^2(\gamma_z - 1)y & -2(\gamma_z + 1)x \end{pmatrix},$$

where $z = x + iy \in \mathbb{C} \setminus \{i, -i/\mu(-1)\}$. Here we have abbreviated

$$\mu = \mu(y) \quad \text{and} \quad \gamma_z = x^2 + \mu(y)^2 y^2 = \|\hat{z}\|^2.$$

References

1. R. Artzy – H. Groh, *Laguerre and Minkowski planes produced by dilatations*, J. Geom. **26** (1986), 1–20.
2. L. Bader – S.E. Payne, *On infinite K-clan geometry*, J. Geom. (to appear).
3. D. Betten, *4-dimensionale Translationsebenen mit 7-dimensionaler Kollineationsgruppe*, J. reine angew. Math. **285** (1976), 126–148.
4. T. Buchanan, *Ovale und Kegelschnitte in der komplexen projektiven Ebene*, Math. Phys. Sem. Ber. **26** (1979), 244–260.
5. T. Buchanan – H. Hähl – R. Löwen, *Topologische Ovale*, Geom. Dedicata **9** (1980), 401–424.
6. M. Forst, *Topologische 4-Gone*, Mitt. Math. Sem. Gießen **147** (1981), 65-129.
7. H. Groh, *Topologische Laguerreebenen I*, Abh. Math. Sem. Univ. Hamburg **32** (1968), 216–231.

8. H. Groh, *Topologische Laguerreebenen II*, Abh. Math. Sem. Univ. Hamburg **34** (1970), 11–21.
9. T. Grundhöfer and N. Knarr, *Topology in generalized quadrangles*, Topology Appl. **34** (1990), 139–152.
10. R. Löwen, *Projectivities and the geometric structure of topological planes*, in: Geometry - von Staudt's point of view (eds P. Plaumann and K. Strambach), Bad Windsheim (1981), 339–372.
11. R. Löwen, *Topological pseudo-ovals, elation Laguerre planes, and elation generalized quadrangles*, Math. Z. **216** (1994), 347–369.
12. R. Löwen and U. Pfüller, *Two-dimensional Laguerre planes over convex functions*, Geom. Dedicata **23** (1987), 73–85.
13. H. Salzmann, D. Betten, T. Grundhöfer, H. Hähl, R. Löwen and M. Stroppel, *Compact projective planes*, de Gruyter, Berlin, 1995.
14. A. Schenkel, *Topologische Minkowskiebenen*, Dissertation, Erlangen-Nürnberg, 1980.
15. A.E. Schroth, *Three-dimensional quadrangles and flat Laguerre planes*, Geom. Dedicata **36** (1990), 365–373.
16. A.E. Schroth, *Generalized quadrangles constructed from topological Laguerre planes*, Geom. Dedicata **46** (1993), 339–361.
17. A.E. Schroth, *Topological antiregular quadrangles*, Result. Math. **24** (1993), 180–189.
18. A.E. Schroth, *Topological circle planes and topological quadrangles*, Pitman Research Notes in Mathematics Series, vol. 337, Longman, Essex, UK, 1995.
19. A.E. Schroth, *Sisterhoods of flat Laguerre planes*, Geom. Dedicata **58** (1995), 185–191.
20. G.F. Steinke, *The automorphism group of locally compact connected topological Benz planes*, Geom. Dedicata **16** (1984), 351-357.
21. G.F. Steinke, *Eine Klassifikation 4-dimensionaler Laguerre-Ebenen mit großer Automorphismengruppe*, Habilitationsschrift, Kiel, 1987.
22. G.F. Steinke, *Topological properties of locally compact connected Minkowski planes and their derived affine planes*, Geom. Dedicata **32** (1989), 341–351.
23. G.F. Steinke, *Semiclassical 4-dimensional Laguerre planes*, Forum Math. **2** (1990), 233–247.
24. G.F. Steinke, *The elation group of a 4-dimensional Laguerre plane*, Monatsh. Math. **111** (1991), 207–231.
25. G.F. Steinke, *4-dimensional Minkowski planes with large automorphism group*, Forum Math. **4** (1992), 593–605.
26. G.F. Steinke, *On 4-dimensional Minkowski planes with 7-dimensional automorphism group*, Geom. Dedicata **48** (1993), 337–348.
27. G.F. Steinke, *Topological circle geometries*, Handbook of incidence geometry (Ed. F. Buekenhout), pp. 1325–1354, Elsevier, Amsterdam, 1995.